



ORIGINAL ARTICLE

Fractional single-phase-lagging heat conduction model for describing anomalous diffusion


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Abstract The fractional single-phase-lagging (FSPL) heat conduction model is obtained by combining scalar time fractional conservation equation to the single-phase-lagging (SPL) heat conduction model. Based on the FSPL heat conduction model, anomalous diffusion within a finite thin film is investigated. The effect of different parameters on solution has been observed and studied the asymptotic behavior of the FSPL model. The analytical solution is obtained using Laplace transform method. The whole analysis is presented in dimensionless form. Numerical examples of particular interest have been studied and discussed in details.

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1. Introduction

The Fourier law of heat conduction assumes that heat flux vector $q(r, t^*)$ and temperature gradient $\nabla T(r, t^*)$ appear at the same time instant t^* and consequently implies

that thermal signal propagates with an infinite speed. The infinite speed of heat propagation, implying that a thermal disturbance applied at a certain location in a medium, can be sensed immediately anywhere else in the medium [1], it is one of the drawbacks of Fourier's law. Many models [2–4] do not agree with the Fourier law since this law is based on infinite speed of heat propagation and simultaneous development of heat flux and temperature gradient.

Cattaneo [5] and Vernotte [6] removed the deficiency of Fourier law by adding a lagging parameter in Fourier law

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Nomenclature

c	thermal wave propagation speed (unit: m/s)
c_b	specific heat capacity (unit: J/(kg · K))
g^*	internal heat generation (unit: W/m ³)
g	dimensionless internal heat generation ($g = 4\alpha_1 g^*/cI_r$)
h	real or complex valued function
I_r	reference heat flux
k	thermal conductivity (unit: W/(m · K))
L	Laplace transform
L^{-1}	inverse Laplace transform
P_d	Predvoditelev number ($P_d = b_1 l^2 / \alpha_1 \Delta T$)
q^*	dimensionless heat flux ($q^* = q/I_r$)
q	heat flux (unit: W/m ²)
\mathbf{r}	position vector
t^*	time (unit: s)
t	dimensionless time ($t = t^*(c^2/2\alpha_1)^{\alpha-1}$)

t_p^*	impulse time (unit: s)
t_p	dimensionless impulse time ($t_p = t_p^*(c^2/2\alpha_1)^{\alpha-1}$)
T	temperature (unit: K)
∇T	temperature gradient (unit: K/m)
x	dimensionless spatial coordinate ($x = cy/2\alpha_1$)
y	spatial coordinate (unit: m)

Greek letters

α_1	thermal diffusivity (unit: m ² /s)
α	order of fractional derivative lies in (0, 1]
θ	dimensionless temperature ($\theta = kcT/\alpha_1 I_r$)
ρ	density (unit: kg/m ³)
τ^*	lagging parameter (unit: s)
τ	dimensionless lagging parameter ($\tau = \tau^*(c^2/2\alpha_1)^{\alpha-1}$)

and proposed the CV constitutive relation in the form of

$$\tau^* \frac{\partial q}{\partial t} + q = -k \nabla T \quad (1)$$

where k is the thermal conductivity of the medium and τ is the material property called the lagging time. This model characterizes the combined diffusion and wave like behavior of heat conduction and predicts a finite speed

$$c = \left(\frac{k}{\rho c_b \tau^*} \right)^{\frac{1}{2}} \quad (2)$$

for heat propagation [7], where ρ is the density and c_b is the specific heat capacity. For more details about thermal lagging in wave theory see [8,9]. Tzou [10–14] generalized the CV heat conduction model [1] as

$$\mathbf{q}(\mathbf{r}, t^* + \tau^*) = -k \nabla T(\mathbf{r}, t^*) \quad (3)$$

The above Eq. (3) is known as the single-phase-lagging (SPL) heat conduction model. This model establishes the heat flux (the result) when a temperature gradient (the cause) is suddenly imposed.

In the past decades, the phenomena of anomalous diffusion have been observed in numerous physical and biological systems [15–22]. The anomalous diffusion is characterized by diffusion constant and the mean square displacement of diffusing species in the form

$$\langle x^2(t) \rangle = t^\alpha, \quad t \rightarrow \infty$$

where α is the diffusion exponent. This phenomena is usually divided into anomalous subdiffusion for $0 < \alpha < 1$ and anomalous superdiffusion for $1 < \alpha < 2$. If $\alpha = 1$, we have the diffusion. To investigate diffusion phenomenon there are numerous approaches have been used [23–26]. Antonakakis et al. [27] analytically solved the diffusion equation with Gaussian heat source. Nishikawa [28] constructed the first, second and third order finite volume scheme for diffusion equation. This method enables straightforward constructions of diffusion schemes for finite-volume methods on unstructured grids.

Now concerning fractional heat transfer, the topic is to some extent new. Povstenko [29] proposed fractional heat equation for modeling thermoelasticity and Ezzat [30] investigated heat transfer in MHD for a thermoelectric medium. Compte and Metzler [31] proposed three possible time fractional generalizations of Cattaneo model, which are supported by continuous-time random walks, non-local transport theory and delayed flux-force relations, respectively. Povstenko [32] studied the time fractional Cattaneo type equations and formulated the corresponding theories of thermal stresses. Qi and Jiang [33] and Atanackovic et al. [34] built the Cattaneo-type space-time fractional heat conduction equation, and the explicit solutions of the Cauchy problem are given in terms of a series and integral representation. Ghazizadeh et al. [35] solved the fractional Cattaneo equation by explicit and implicit finite difference schemes. Qi et al. [36] are used the generalized Cattaneo model with fractional derivative and solved the corresponding Cattaneo-type fractional heat conduction equation for laser heating by Laplace transforms technique. Very recently, Ghazizadeh et al. [37] generalized the SPL heat conduction model to the FSPL heat conduction model. They obtained the FSPL heat conduction model by applying fractional Taylor series formula [38] on the SPL delayed equation.

In the present study, the fractional single-phase-lagging (FSPL) heat conduction model is obtained by combining the SPL model to the fractional conservation equation and studied the combined effects of anomalous subdiffusion and anomalous superdiffusion. The effects of fractional order (α) and heat source (g) on temperature distributions within the thin film based on the FSPL heat conduction model will also be investigated and studied the asymptotic behavior (i.e. $\alpha \rightarrow 1$) of the FSPL model. The outline of the paper is as follows. In Section 2, FSPL heat conduction model is presented. Solution is given in Section 3. Section 4 contains results and discussion. Concluding remarks are summarized in Section 5.

2. FSPL heat conduction model

The combination of Fourier law of heat conduction

$$q = -k \frac{\partial T}{\partial y} \quad (4)$$

and scalar time fractional conservation equation [35]

$$\rho c_b \frac{\partial^\alpha T}{\partial t^{\alpha}} = -\frac{\partial q}{\partial y} + g^*, \quad 0 < \alpha \leq 1 \quad (5)$$

provides the fractional heat conduction equation as follows

$$\rho c_b \frac{\partial^\alpha T}{\partial t^{\alpha}} = k \frac{\partial^2 T}{\partial y^2} + g^*, \quad 0 < \alpha \leq 1 \quad (6)$$

where g^* denotes the internal energy generation rate per unit volume inside a medium. Above Eq. (6) is the fractional diffusion model which governs thermal energy transport in solids. Now by combining one dimensional form of Eq. (1) with the scalar time fractional conservation Eq. (5) we get the FSPL equation

$$\frac{1}{\alpha_1} \frac{\partial^\alpha T(y, t^*)}{\partial t^{\alpha}} + \frac{\tau^* \partial^{1+\alpha} T(y, t^*)}{\alpha_1 \partial t^{*(1+\alpha)}} = \frac{\partial^2 T(y, t^*)}{\partial y^2} + \frac{1}{k} \left(g^* + \tau^* \frac{\partial g^*}{\partial t^*} \right), \quad 0 < \alpha \leq 1 \quad (7)$$

based on the Caputo definition. Here α_1 is the thermal diffusivity of the material. By introducing dimensionless parameters,

$$\theta = \frac{kcT}{\alpha_1 I_r}, \quad x = \frac{cy}{2\alpha_1}, \quad t = \left(\frac{c^2}{2\alpha_1} \right)^{\frac{1}{\alpha}} t^*,$$

$$\tau = \left(\frac{c^2}{2\alpha_1} \right)^{\frac{1}{\alpha}} \tau^*, \quad \text{and} \quad g = \frac{4\alpha_1 g^*}{cI_r}$$

Eq. (7) can be expressed in dimensionless form as follows

$$2 \left(\frac{\partial^\alpha \theta(x, t)}{\partial t^\alpha} + \tau \frac{\partial^{1+\alpha} \theta(x, t)}{\partial t^{(1+\alpha)}} \right) = \frac{\partial^2 \theta(x, t)}{\partial x^2} + \left(g + \tau \frac{\partial g}{\partial t} \right), \quad 0 < \alpha \leq 1 \quad (8)$$

Setting $\alpha = 1$ in Eqs. (6) and (8) results to standard diffusion equation and the SPL equation, respectively. For $0 < \alpha < 1$ the first and second terms of the left hand side of Eq. (8) correspond for the anomalous subdiffusion and anomalous superdiffusion, respectively. For $\alpha \rightarrow 0$, the FSPL heat conduction Eq. (8) asymptotically similar to the fractional diffusion model.

In present study, an isotropic thin film heated by an internal heat source (g), confined between $0 \leq x \leq 1$, with uniform thickness and constant thermophysical properties are assumed. Initially thin film and temporal temperature variation within the thin film are at same temperature $f_1(x) = \sin(\pi x)$, which is a function of position within

the solid. The ends ($x = 0$) and ($x = 1$) of the thin film are at the same temperature $g_1 = -2g$. Thus the initial and boundary conditions for heat conduction in thin film are

$$\theta(x, 0) = f_1(x), \quad \frac{\partial \theta(x, 0)}{\partial t} = f_1(x) \quad (9)$$

$$\theta(0, t) = g_1(t) \quad (10)$$

$$\theta(1, t) = g_1(t) \quad (11)$$

Note that the initial and boundary conditions were arbitrarily selected in this study and may not be a common one used in most heat transfer investigations. The intent of this study is to demonstrate the versatility and benefits of the model rather than investigate a particular application. However, the solution method is capable of applying conventional conditions (e.g., zero or other types of initial conditions) without any difficulties but it is may be difficult to apply in the case of most generalized boundary conditions (second/third or mixed) as in present study the solution method is exact.

3. Solution

Laplace transform of Caputo fractional derivative [20] can be written as

$$\bar{h}(s) = L \left[\frac{\partial^\alpha h(t)}{\partial t^\alpha} \right] = s^\alpha \bar{h}(s) - s^{\alpha-1} h(0), \quad 0 < \alpha \leq 1$$

Case I: Assume that thin film is heated by a exponentially decaying heat source [39–41],

$$g = \frac{1}{2} \left(1 - e^{-P_d \left(\frac{t}{t_p} \right)} \right) \quad (12)$$

where t_p is the impulse time and P_d is the Predvoditelev number. Applying Laplace transform to Eqs. (8)–(11) produces

$$\frac{d^2 \bar{\theta}(x, s)}{dx^2} = 2(s^\alpha \bar{\theta}(x, s) - s^{\alpha-1} \theta(x, 0)) - \bar{g}(x, s)(1 + \tau) + g(x, 0) + 2\tau \left(s^{1+\alpha} \bar{\theta}(x, s) - s^\alpha \theta(x, 0) - s^{\alpha-1} \frac{\partial \theta(x, 0)}{\partial t} \right) \quad (13)$$

$$\theta(x, 0) = \sin(\pi x), \quad \frac{\partial \theta(x, 0)}{\partial t} = \sin(\pi x) \quad (14)$$

$$\bar{\theta}(0, s) = \bar{g}_1(s), \quad \bar{\theta}(1, s) = \bar{g}_1(s) \quad (15)$$

On solving Eqs. (12)–(15), we get

$$\bar{\theta}(x, s) = \frac{(2 + 2\tau) s^{\alpha-1} \sin(\pi x)}{\pi^2 + 2s^\alpha + 2\tau s^{1+\alpha}} + \frac{2s^\alpha \sin(\pi x)}{\pi^2 + 2s^\alpha + 2\tau s^{1+\alpha}} - \frac{(1 + \tau)}{2} \left(\frac{1}{s} - \frac{1}{s + \frac{P_d}{t_p}} \right) \quad (16)$$

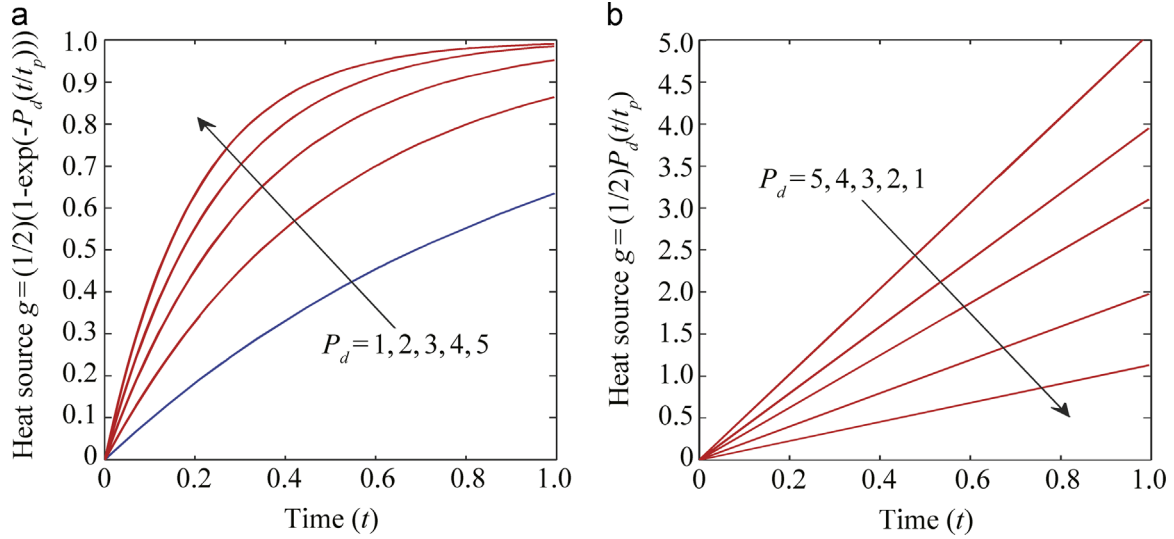


Figure 1 Variation of heat sources with time.

Now taking $\tau = 1$, Eq. (16) becomes

$$\bar{\theta}(x, s) = \frac{4s^{\alpha-1} \sin(\pi x)}{\pi^2 + 2s^\alpha + 2s^{1+\alpha}} + \frac{2s^\alpha \sin(\pi x)}{\pi^2 + 2s^\alpha + 2s^{1+\alpha}} - \left(\frac{1}{s} - \frac{1}{s + \frac{P_d}{t_p}} \right) \quad (17)$$

To find the inverse Laplace transform, first we re-write Eq. (17) as

$$\bar{\theta}(x, s) = \frac{4 \sin(\pi x)}{\pi^2 s^{\alpha-1} + 2s + 2s^2} + \frac{2 \sin(\pi x)}{\pi^2 s^{-\alpha} + 2 + 2s} - \left(\frac{1}{s} - \frac{1}{s + \frac{P_d}{t_p}} \right) \quad (18)$$

Simplifying Eq. (18) we have

$$\bar{\theta}(x, s) = \left(1 + \frac{2}{s} \right) \frac{\sin(\pi x)}{s} \left[1 + \left(\frac{1}{s} + \frac{n\pi^2}{s^{1+\alpha}} \right) \right]^{-1} - \left(\frac{1}{s} - \frac{1}{s + \frac{P_d}{t_p}} \right) \quad (19)$$

where $n = \frac{1}{2}$. Now taking inverse Laplace transform of above Eq. (19) we get

$$\theta(x, t) = 2 \sin(\pi x) L^{-1} \left[\frac{1}{s^2} \sum_{p=0}^{\infty} (-1)^p \left(\frac{1}{s} + \frac{n\pi^2}{s^{1+\alpha}} \right)^p \right] + \sin(\pi x) L^{-1} \left[\frac{1}{s} \sum_{p=0}^{\infty} (-1)^p \left(\frac{1}{s} + \frac{n\pi^2}{s^{1+\alpha}} \right)^p \right] - \left(1 - e^{-P_d \left(\frac{t}{t_p} \right)} \right) \quad (20)$$

Case II: Assume that thin film is heated by a linear heat source [39–41],

$$g = \frac{1}{2} P_d \left(\frac{t}{t_p} \right) \quad (21)$$

Solving Eqs. (13)–(15) and using Eq. (21) we get

$$\bar{\theta}(x, s) = \frac{(2 + 2\tau)s^{\alpha-1} \sin(\pi x)}{\pi^2 + 2s^\alpha + 2\tau s^{1+\alpha}} + \frac{2s^\alpha \sin(\pi x)}{\pi^2 + 2s^\alpha + 2\tau s^{1+\alpha}} - \frac{(1 + \tau)}{2} \left(\frac{P_d}{t_p s^2} \right) \quad (22)$$

For $\tau = 1$, solution of above Eq. (22) becomes

$$\theta(x, t) = 2 \sin(\pi x) L^{-1} \left[\frac{1}{s^2} \sum_{p=0}^{\infty} (-1)^p \left(\frac{1}{s} + \frac{n\pi^2}{s^{1+\alpha}} \right)^p \right] + \sin(\pi x) L^{-1} \left[\frac{1}{s} \sum_{p=0}^{\infty} (-1)^p \left(\frac{1}{s} + \frac{n\pi^2}{s^{1+\alpha}} \right)^p \right] - P_d \left(\frac{t}{t_p} \right) \quad (23)$$

In the above solution, Mathematical software ATHEMATICA 7.0 has been used to obtain dimensionless temperature θ .

4. Results and discussion

This section presents complete analysis of thermal propagation within a finite thin film based on the FSPL heat conduction model. Results are presented to demonstrate the effects of fractional order parameter (α) and lagging parameter (τ) on temperature profiles within the medium. The asymptotic behavior of the FSPL model has also been studied. The parameters whose values different from the reference value are indicated. The selected reference values include lagging parameter $\tau = 1$ and impulse time $t_p = 1$.

Figure 1 presents the temporal profile of heat sources for different Predvoditelev number (P_d). As heat source is a function of Predvoditelev number (P_d) and time t and if P_d increases and tends to infinity at any time $t \rightarrow 0$ then $\exp(-P_d t)$ decreases and tends to zero, hence heat source (g) $\rightarrow 1$ i.e. constant heat source, as shown in Figure 1(a).

Figure 1(b) evidently shows that heat source increases linearly with increase of P_d or t .

Case I: Thin film is heated by a exponentially decaying heat source $g = \frac{1}{2} \left(1 - e^{-P_d \left(\frac{t}{t_p} \right)} \right)$

Figures 2 and 3 show that the spatial temperature distribution of thin film with Predvoditelev number (P_d) for different fractional order (α). For fractional order $0 < \alpha < 1$, anomalous subdiffusion and anomalous superdiffusion phenomena occur simultaneously. But in the case of small heating period i.e. $t = 0.5$, anomalous superdiffusion dominates the anomalous subdiffusion, consequently temperature increases with increase of α . For $\alpha = 1$ temperature increases rapidly and attains a maximum value and then decreases symmetrically showing that the hyperbolic feature. A close examination of Figures 2 and 3 also reveal that, if P_d increases then amount of heat entering into the body increases, consequently temperature increases.

Temporal temperature distributions of thin film with P_d for different fractional orders (α) are shown in Figures 4 and 5. For small fractional order (α) as given in Figure 4(a), temperature increases rapidly in small change of time and

large P_d which exhibits the anomalous superdiffusion phenomenon. As P_d decreases, the amount of heat entering into the body decreases, hence rate of increase of temperature is slower in case of small P_d .

From Figures 4(b)–5(a), it is evident that, for small P_d , temperature increases and attains a maximum temperature and then decreases very slowly with time, which shows the combined effect of anomalous diffusion and anomalous superdiffusion. Further, increasing effect of P_d on temperature profile ceases the decreasing behavior of temperature profile with time. From Figure 5(b), it is observed that temperature increases rapidly after attaining maximum temperature in small time period then decreases gradually, which exhibits the hyperbolic behavior of temperature profile.

Figure 6 presents the limiting behavior of the FSPL model exhibits the diffusion like feature and hence temperature of the thin film decreases and attains a stable state. Further, Figure 6(b) shows the validity of model, as in absence of heat source, temperatures at the boundaries are zero.

Figure 7 represents the temporal temperature profile at mid point ($x = 0.5$) of thin film. For very small fractional order ($\alpha \rightarrow 0$) there are two cases arises, one is the case of small

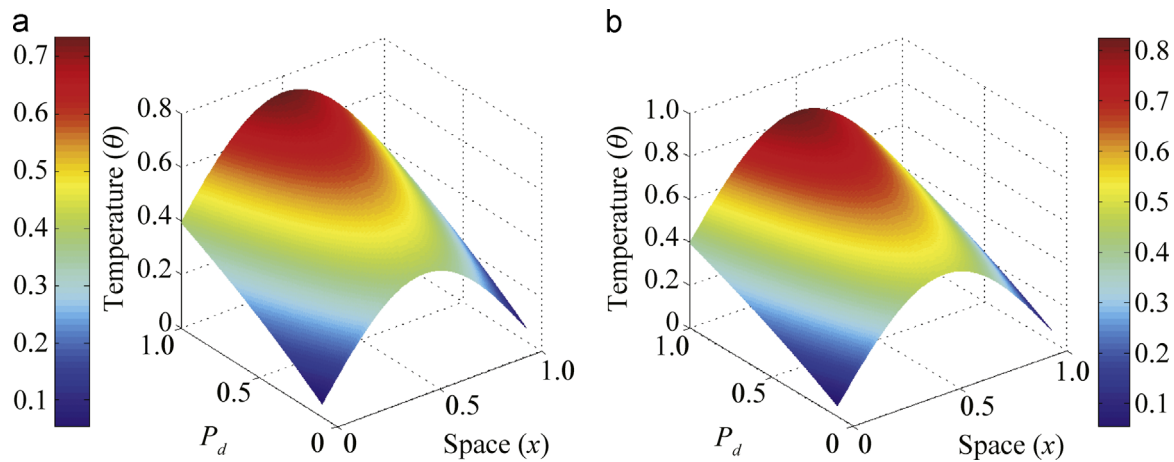


Figure 2 Spatial temperature profile at $t = 0.5$ for (a) $\alpha = 0.1$ and (b) $\alpha = 0.4$.

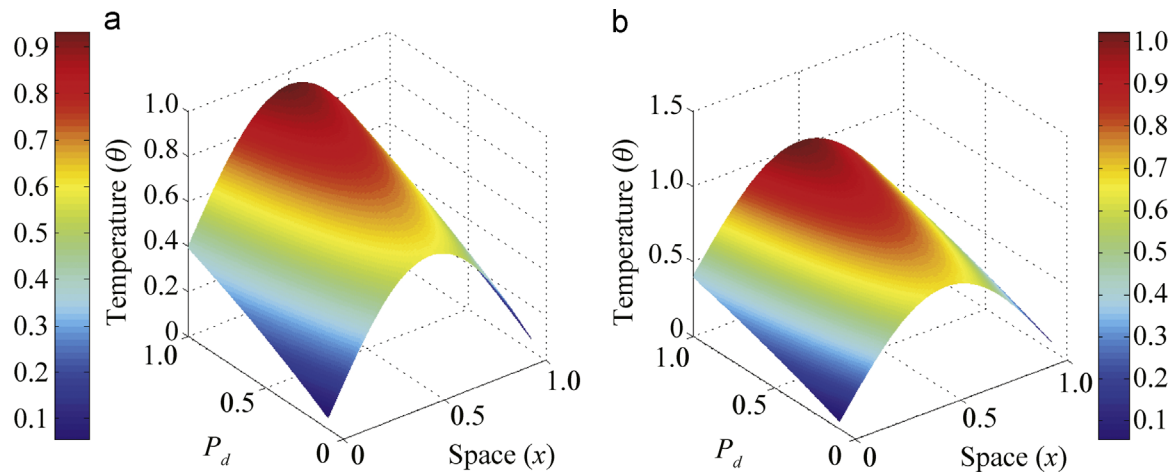


Figure 3 Spatial temperature profile at $t = 0.5$ for (a) $\alpha = 0.7$ and (b) $\alpha = 1.0$.

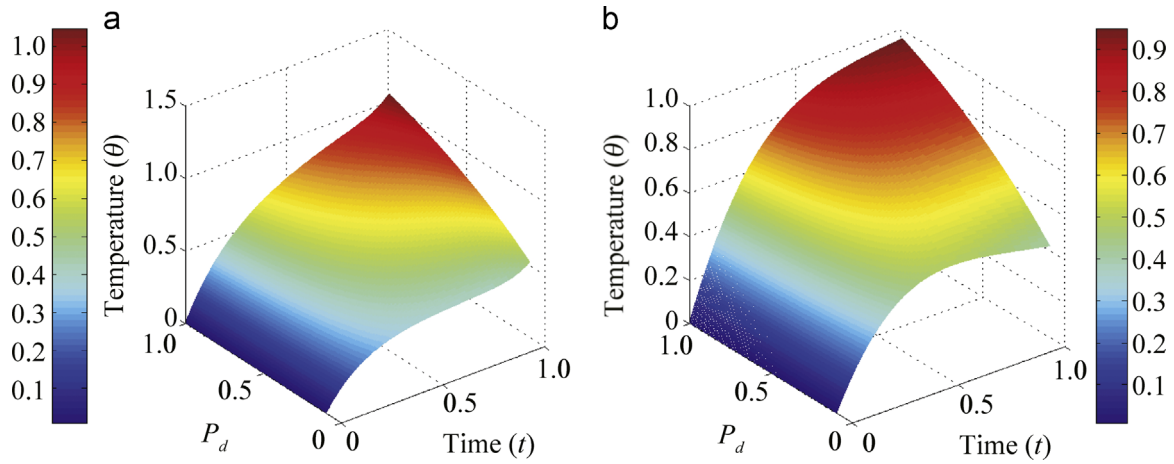


Figure 4 Temporal temperature profile at $x=0.5$ for (a) $\alpha=0.1$ and (b) $\alpha=0.4$.

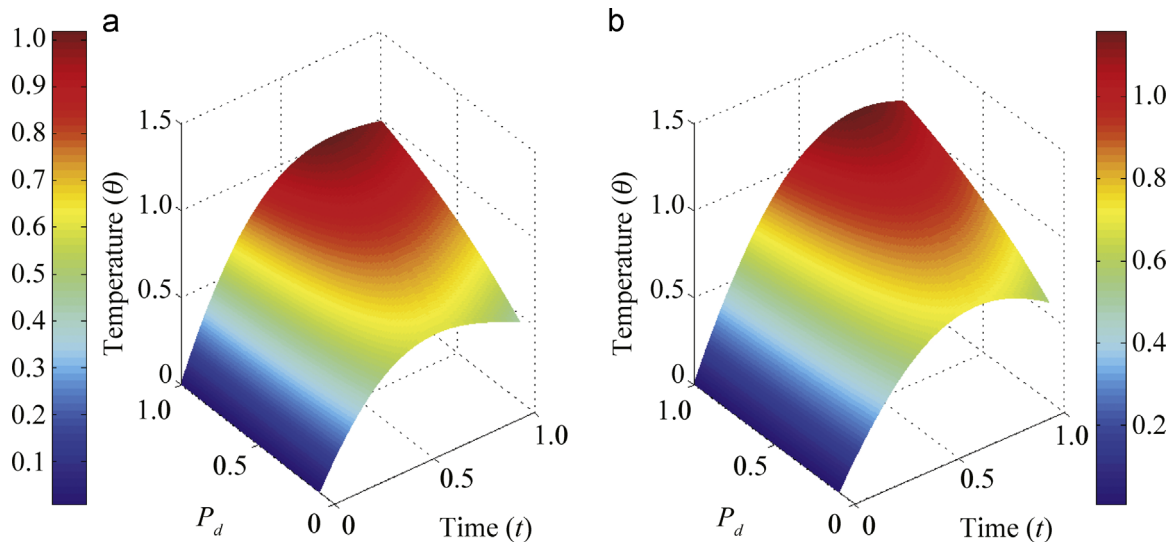


Figure 5 Temporal temperature profile at $x=0.5$ for (a) $\alpha=0.7$ and (b) $\alpha=1.0$.

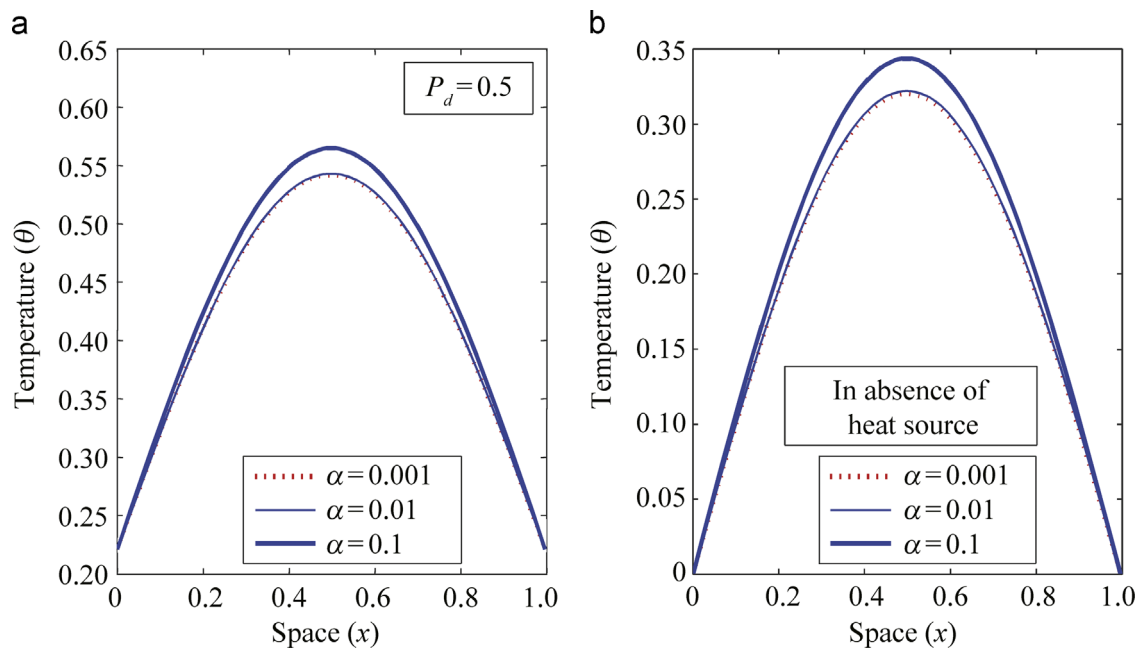
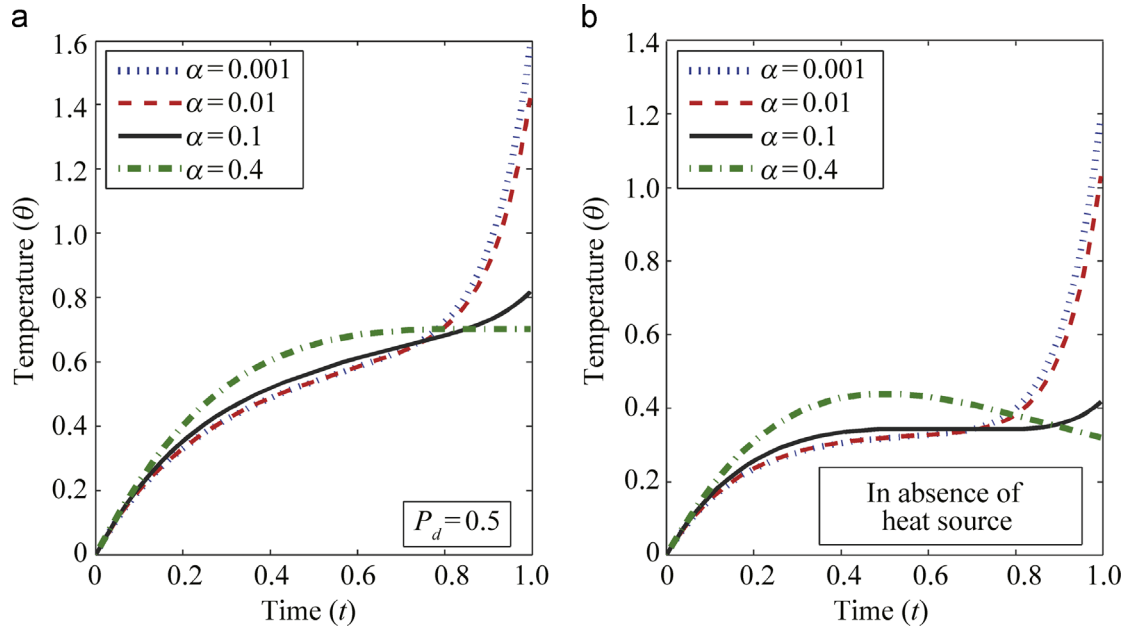
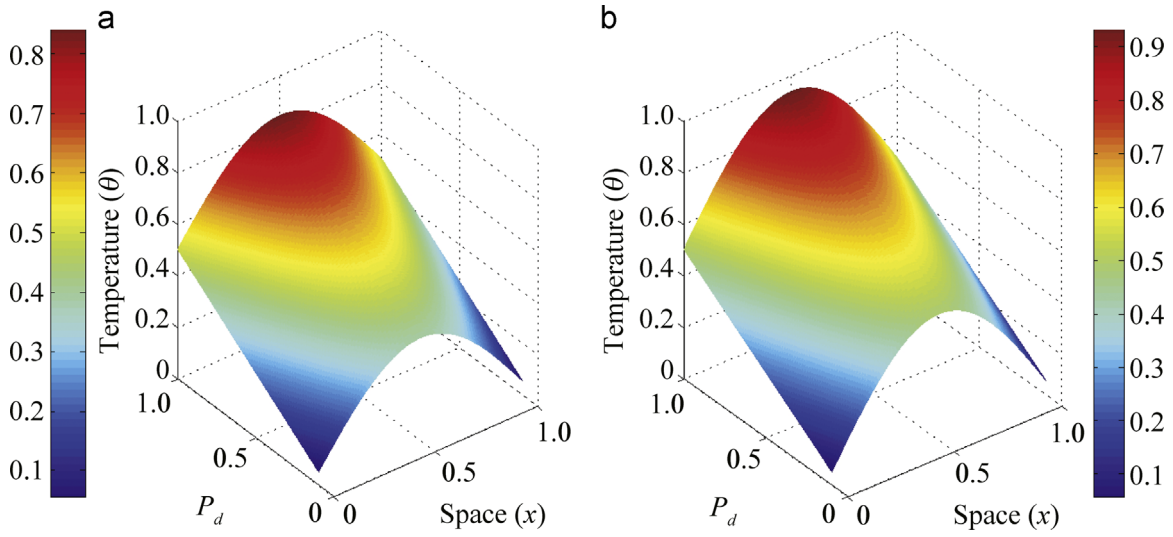
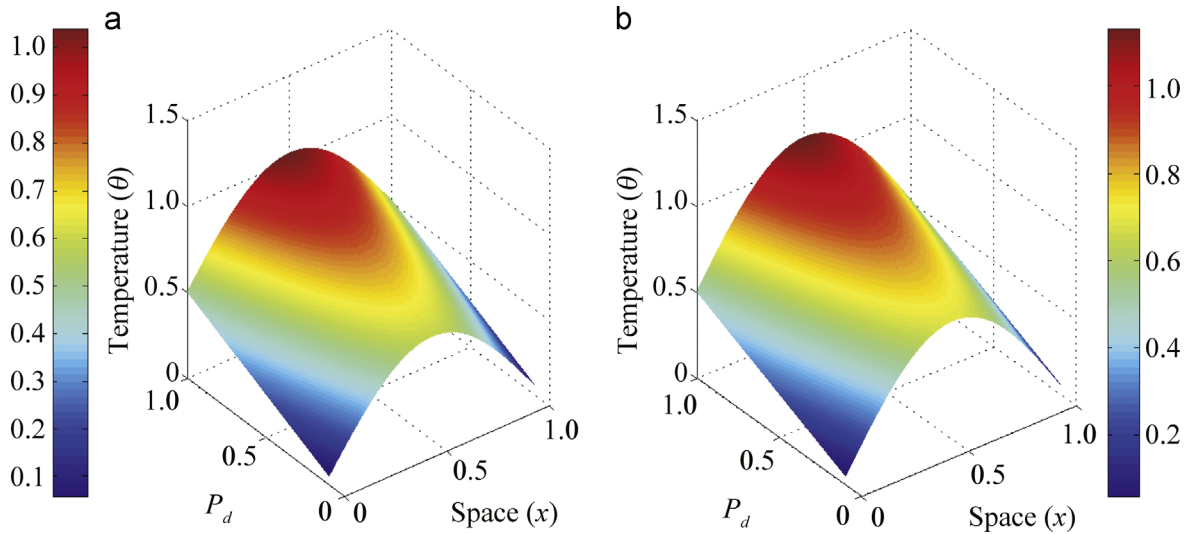


Figure 6 Spatial temperature profile at $t=0.5$.


 Figure 7 Temporal temperature profile at $x=0.5$.

 Figure 8 Spatial temperature profile at $t=0.5$ for (a) $\alpha=0.1$ and (b) $\alpha=0.4$.

 Figure 9 Spatial temperature profile at $t=0.5$ for (a) $\alpha=0.7$ and (b) $\alpha=1.0$.

heating period where temperature increases with increase of α , as in this case the FSPL model asymptotically changes to the diffusion model. In the case of large heating period, due to the effect of heat source, temperature increases rapidly with time, Figure 7(a). Further, the increasing behavior of temperature profile is gradual in the absence of heat source, as shown in Figure 7.

Case II: Thin film is heated by a linear heat source $g = \frac{1}{2}P_d\left(\frac{t}{t_p}\right)$

Figures 8 and 9 show that spatial temperature distribution of thin film with P_d , for different fractional orders (α). In this case heat source (g) is directly proportional to P_d and t , due to which the rate of entering heat into the body at any time is more than that of Case I, (Figures 2 and 3), consequently rate of increase of temperature is more than that of Case I. A close examination of Figures 8 and 9 together with Figures 2 and 3 shows that the effect of P_d on

temperature profile is similar to the results of Figures 2 and 3 but in this case due to the linear heat source, rate of increase of temperature is more than that of Figures 2 and 3.

Temporal temperature distributions of thin film with P_d for different fractional orders (α) are shown in Figures 10 and 11. As expected from the given heat source, (Figure 1 (b)), temperature increases rapidly with time and P_d . A close examination of Figures 10 and 11, together with Figures 4 and 5 reveals that, temperature of body is higher than that of Case I, due to the effect of linear heat source.

Figure 12 presents the limiting behavior of the FSPL heat conduction model in the presence and absence of heat source. In the presence of heat source (Figure 12(a)) and at same $P_d = 0.5$ and time (t) = 0.5, the temperature profile is similar to the spatial temperature profile of Figure 6(a), but in this case due to the effect of linear heat source rate of increase of temperature is increases as α increases. In the

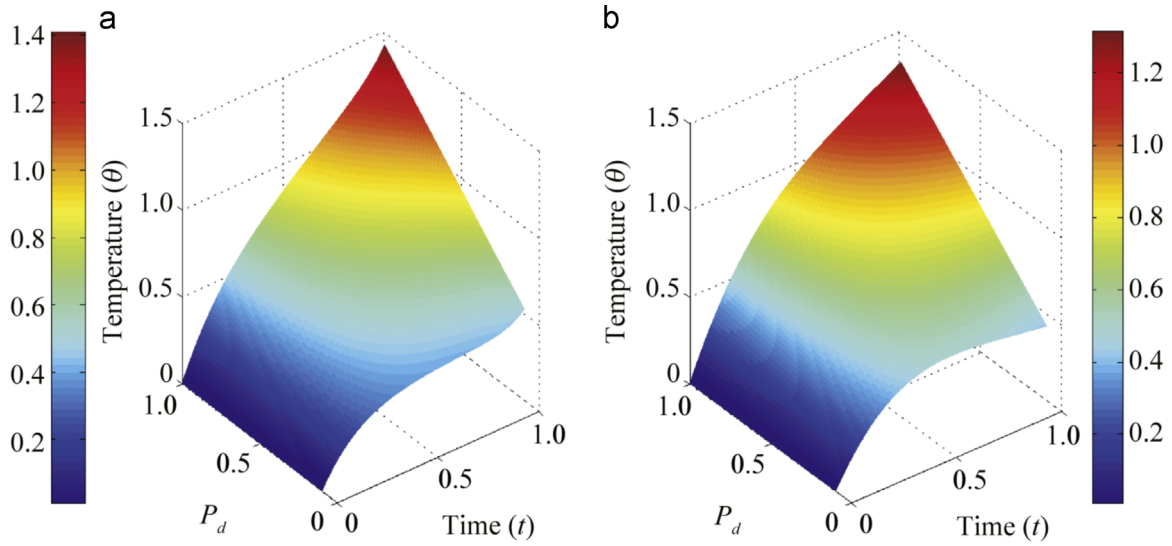


Figure 10 Temporal temperature profile at $x=0.5$ for (a) $\alpha=0.1$ and (b) $\alpha=0.4$.

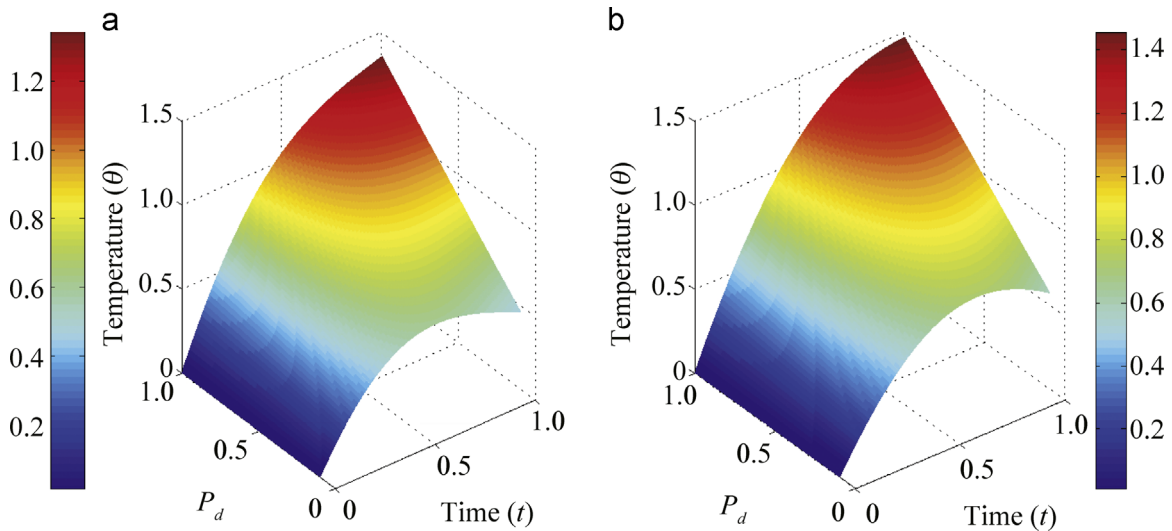


Figure 11 Temporal temperature profile at $x=0.5$ for (a) $\alpha=0.7$ and (b) $\alpha=1.0$.

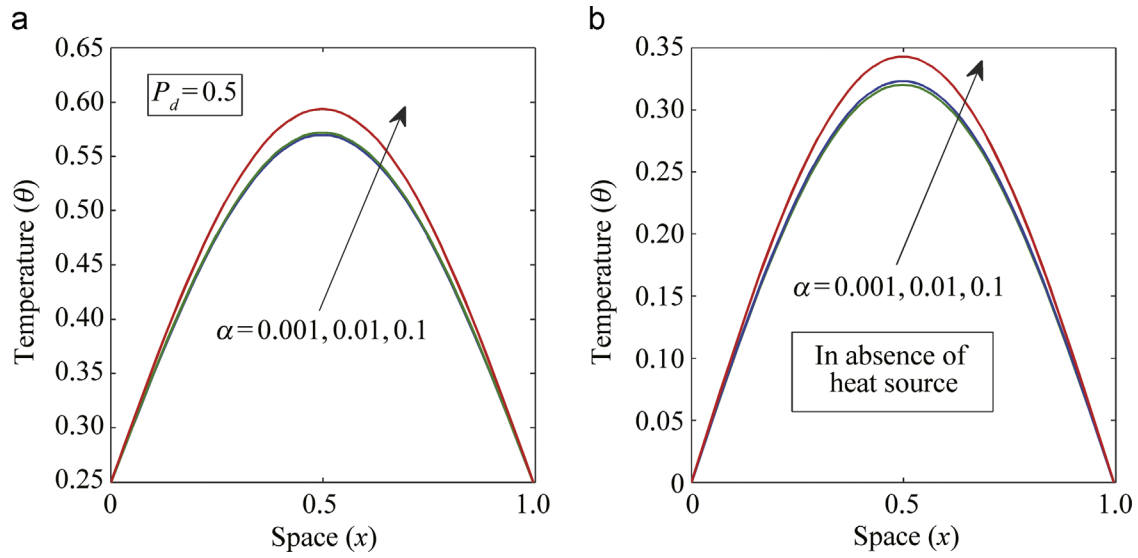


Figure 12 Spatial temperature profile at $t = 0.5$.

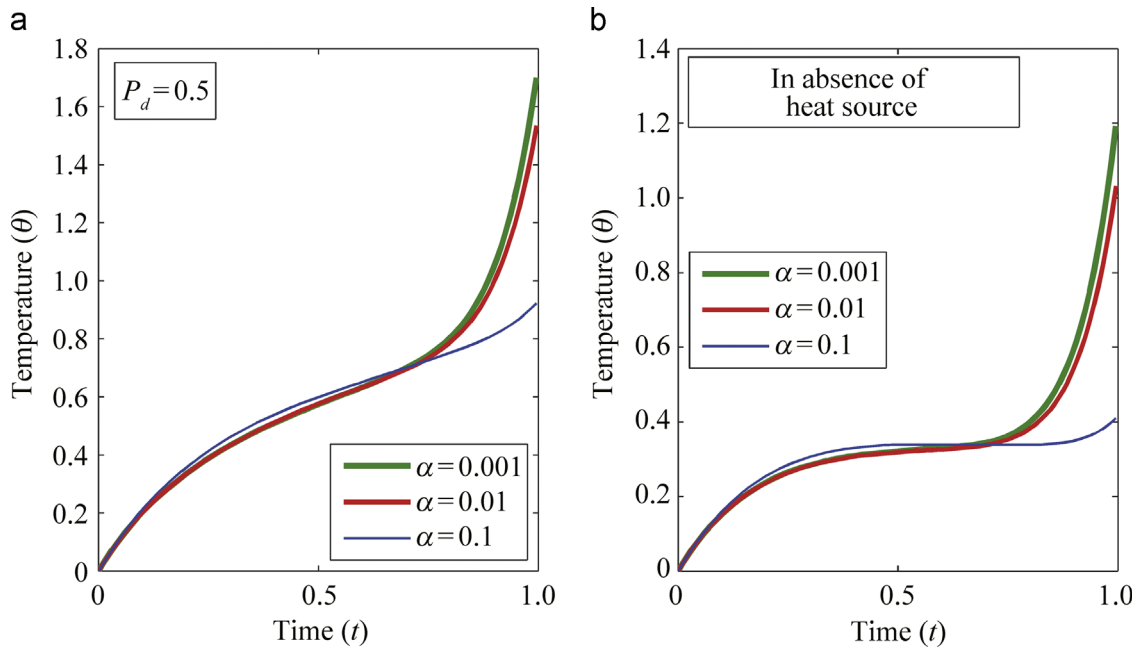


Figure 13 Temporal temperature profile at $x = 0.5$.

absence of heat source, Figures 6(b) and 12(b) show the same temperature profiles.

A close examination of Figure 12(a) and (b) reveals that linear heat source creates the large amount of heat into the body as a result of which temperature of body in the presence of heat source is more than the temperature of body in the absence of heat source.

Figure 13 shows that the temporal temperature profiles at midpoint ($x = 0.5$) of the thin film. At fixed P_d and in the presence of heat source, a fixed amount of heat entered into the body. This heat increases with increase of time, due to which temperature of the body is greater than the temperature of body in the absence of heat source for corresponding fractional order (α).

5. Conclusion

A mathematical model of the FSPL heat conduction based on fractional conservation equation is solved by Laplace transform technique. Results were presented in this manuscript to demonstrate the effects of fractional order parameter (α) and heat source (g) on the temperature distributions within the thin film. In the case of $0 < \alpha < 1$, the FSPL heat conduction model interpolates the anomalous subdiffusion and anomalous superdiffusion, but for very small heating period anomalous superdiffusion dominated the anomalous subdiffusion. For $\alpha \rightarrow 0$, the FSPL model asymptotically changes to the diffusion model and if $\alpha = 1$, the FSPL model completely change to the SPL (hyperbolic) heat conduction model. The linear heat source

produces more heat into the body as compared to the exponentially decaying heat source. This model is a generalization of classical heat conduction model, anomalous diffusion model and SPL model.

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